Bayesian Approach for Recursive Compressed Sensing

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***Abstract*—We consider the problem of recursively reconstructing a sequence of sparse signals using compressed sensing measurements. We assume that each consecutive signal vectors of the sparse signal sequence are partially overlapped. Our aim in this work is to propose an effective recursive method for the reconstruction of the sequence of signals based on Bayesian compressed sensing.**

I. Introduction

In many systems in science and engineering, a real-valued signal vector, $\mathbf{x} \in \mathbb{R}^N$ indexed as $x_n, n \in \{1, 2, ..., N\}$, is measured via linear operator

$$\mathbf{y} = \mathbf{\Phi x} + \mathbf{z}, \tag{1}$$

where $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$ denotes the measurement matrix and $\mathbf{z} \in \mathbb{R}^{\mathbf{M}}$ denotes the additive noise. The goal is to estimate $\mathbf{x}$ from the measurements $\mathbf{y}$, when $\mathbf{\Phi}$ and a statistical distribution for the noise $\mathbf{z}$ is given. This estimation problem is commonly referred to as recovery or reconstruction problem.

When $M < N$, the setup is known as compressed sensing (CS) and the inverse problem is ill-posed since the number of measurements is smaller than the degrees of freedom of the signal $\mathbf{x}$. However, if the signal $\mathbf{x}$ is sufficiently sparse it can still be accurately estimated from an underdetermined set of measurements [1], [2]. A typical means of solving such an ill-posed problem, for which it is known that $\mathbf{x}$ is sparse and noise is Gaussian, is via an l_1-regularized formulation [3]

$$\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \{\|\mathbf{x}\|_1 + \gamma\|\mathbf{y} - \mathbf{\Phi x}\|_2^2\} \tag{2}$$

where the scalar γ controls the relative importance applied to the sparseness term and the Euclidian error (the first and second expressions inside the brackets in (2), respectively). This results in an estimation of $\mathbf{x}$ and this basic framework has been the starting point for several CS inversion algorithms.

In most of the real-world scenarios, time sequences of sparse signals have to be reconstructed recursively from linear measurements [4], [5]. Each time a measurement vector is obtained a recovery problem should be solved from scratch, since the conventional CS approach assumes no prior information on the unknown signal $\mathbf{x}$ other than the fact that it is sufficiently sparse in a particular basis. However, in many applications additional prior information is available and such information can be used to design efficient CS algorithms. Modified CS recovery algorithms have been proposed for partially known support of the signal $\mathbf{x}$ in [4] and probabilistic prior on the support in [6]. In [5] a *homotopy continuation* method and in [7] a recursive algorithm have been proposed to speed up the current optimization using past estimates.

In this work, we consider the problem of recursively reconstructing time sequences of sparse signals. Our aim is to use the available prior knowledge on the signals and propose an efficient method for reconstruction of the sequence of sparse signals using Bayesian compressed sensing (BCS).

II. Bayesian compressed sensing

In BCS, the inversion of the compressive measurements obtained from (1) is considered from a Bayesian perspective [8]. There, we have a prior belief that $\mathbf{x}$ should be sparse, $\mathbf{y}$ is observed from compressive measurements, and the objective is to provide a posterior belief for the values of the elements of $\mathbf{x}$. For given measurements $\mathbf{y}$, the *posteriori distribution* of $\mathbf{x}$ can be written as

$$f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y}) = \frac{f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) f_{\mathbf{x}}(\mathbf{x})}{\int f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) f_{\mathbf{x}}(\mathbf{x})\ dx}. \tag{3}$$

By using (3), BCS can provide a full posterior density function of $\mathbf{x}$, which provide more information on the estimation of $\mathbf{x}$, rather than providing a point (single) estimate.

A point estimate can be obtained using the maximum *a posteriori* (MAP) estimator. The MAP estimator for $\mathbf{x}$ has the form

$$\mathbf{x}_{\text{map}} = \arg\max_{\mathbf{x}} f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}). \tag{4}$$

Let the entries of the additive noise $\mathbf{z}$ to be i.i.d Gaussian with zero mean and known variance σ^2. Then the *likelihood function*

$$f_{\mathbf{y}|\mathbf{x}}(\mathbf{y}|\mathbf{x}) = c_1 e^{-c_2\|\mathbf{y}-\mathbf{\Phi x}\|_2^2}\ , \tag{5}$$

where $c_1 = (2\pi\sigma^2)^{-M/2}$ and $c_2 = 1/2\sigma^2$ are constants.

We can place a sparseness-promoting prior on $\mathbf{x}$ to model the sparseness in $\mathbf{x}$. A widely used sparseness prior is the Laplace density function [8]

$$f_{\mathbf{x}}(\mathbf{x}|\lambda) = \left(\frac{\lambda}{2}\right)^N \exp\left(-\lambda \sum_{n=1}^{N} |x_i|\right) = c_3 e^{-c_4\|\mathbf{x}\|_1} \tag{6}$$

where $c_3 = \left(\frac{\lambda}{2}\right)^N$ and $c_4 = \lambda$ are constants, and λ is a parameter associated with the density function (large λ corresponds to high probability of having a zero). Then (4) reduce to

$$\mathbf{x}_{\text{map}} = \arg\min_{\mathbf{x}} \quad \left\{c_4\|\mathbf{x}\|_1 + c_2\|\mathbf{y} - \mathbf{\Phi x}\|_2^2\right\}, \tag{7}$$

which is equivalent to l_1-regularized formulation (2). This implies that the MAP estimation of $\mathbf{x}$ with Laplace spareness prior is equivalent to conventional CS inversion with l_1-regularized formulation.

III. System model and problem formulation

In this work, we consider the problem of recursively reconstructing a sequence of sparse signals using CS measurements. Specifically, we investigate the CS recovery of the sequence of sparse signals when the signal sequence has the following structure (see Figure 1).

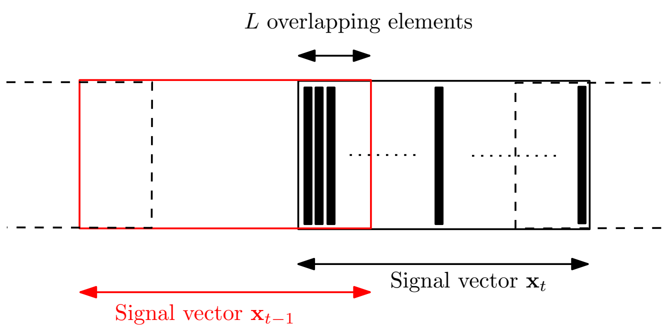

Fig. 1. Signal structure. Each signal vector x_t consists of N real valued elements. Each two consecutive signal vectors are overlapped each other.

We consider a sequence of sparse signals where each two consecutive signal vectors are overlapped each other, i.e., the last L elements of the sparse signal vector $\mathbf{x_{t-1}}$ are similar to the first L elements of signal vector $\mathbf{x}_t$. We assume that at time slot t a real-valued signal vector, $\mathbf{x}_t \in \mathbb{R}^N$, is measured via linear operator

$$\mathbf{y}_t = \mathbf{\Phi}\mathbf{x}_t + \mathbf{z}_t, \quad t = 1, 2, ... \tag{8}$$

where $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$ denotes the measurement matrix and $\mathbf{z}_t \in \mathbb{R}^M$ denotes the additive noise with entries i.i.d Gaussian with zero mean and known variance σ^2. We assume that the reconstruction is done recursively, hence signal $\mathbf{x}_{t-1}$ is reconstructed before the reconstruction of signal $\mathbf{x}_t$.

IV. Solution approach

Consider the problem of reconstructing signal $\mathbf{x}_t$. Besides, the prior information that the signal vector is sparse, we have prior information on the first L elements of signal vector $\mathbf{x}_t$, since they have been already reconstructed when signal $\mathbf{x}_{t-1}$ is recovered.

The conventional CS approach would be to recover the signal $\mathbf{x}_1$ using a CS recovery method[1]. Then recover each signal $\mathbf{x}_t$ using a CS recovery method while setting the first L elements of the signal vector to previously estimated values. This method does not take in to account any recovery errors, hence the error propagates through the reconstruction.

Our aim is to propose a BCS based approach for recovery. As mentioned in Section II, BCS provides the full *posterior distribution* of the signal, which yields the "error bars" on the estimated signal. Hence, we can use the estimate and also the error bars of $\mathbf{x}_{t-1}$ as prior knowledge in recovering $\mathbf{x}_t$.

[1]No prior information about $\mathbf{x}_1$ is available other than the signal is sparse.

V. Conclusion

We have considered the problem of recursively reconstructing a sequence of sparse signals using compressed sensing measurements. Each consecutive signal vectors of the sparse signal sequence are assumed to be partially overlapped. We proposed an effective recursive method for the reconstruction of the sequence of signals based on Bayesian compressed sensing.

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