Effect of Diversity on Estimation Error Outage for Scalar Gauss-Markov Models Sent Over Fading Channels

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***Abstract*—A scalar first-order Gauss-Markov process is sent over d parallel i.i.d. Rayleigh fading channels using uncoded analog transmission. With known channel realization at the decoder, the optimal MMSE estimator is the Kalman filter with random instantaneous estimation error variance. Statistical characteristics of the instantaneous estimation error variance and especially the estimation error outage are studied to characterize the estimation quality. We study the effect of channel diversity on the estimation error outage and show that for a certain range of outage thresholds, the estimation error outage decreases with d-th power of the average channel SNR.**

***Index Terms*—Estimation over Fading Channels, Diversity, Outage, Kalman Filter**

I. Introduction

Uncoded analog transmission of Gauss-Markov sources over fading channels is of interest in a variety of real-time communication systems where such scheme can meet the tough requirements on delay and offer low complexity. Reconstruction at the receiver may be achieved using low complexity linear filtering schemes and optimal MMSE reconstruction is made possible using the Kalman filter. The Kalman filter gain and the instantaneous estimation error variance (IEEV) have then random values. The statistical characteristics of the IEEV are required in order to evaluate the quality of the reconstructed signal at the receiver.

Kalman filtering with random parameters ([1]–[3]) results in a random IEEV whose statistical characteristics, such as mean and pdf are of interest to characterize estimation quality and to determine stability for control purposes. The issue of stability was considered in [4] while in [5], the peak covariance stability of the estimation error covariance matrix resulting from Kalman filtering with random observation losses is studied. Also in [6] bounds on the mean of the instantaneous covariance matrices with some random Riccati formulation are obtained. Using the random matrix theory framework, the stationary distributions for infinitely large random matrices were also found in [7] and [8] for two classes of random Riccati and Lyapunov equations. In this work, we have targeted the outage probability for IEEV as another measure for estimation quality assessment most useful for applications with tight delay requirements.

We focus on the case where first-order Gauss-Markov signals are sent over d parallel i.i.d Rayleigh fading channels. The use of d channels is for additional low latency reliability, when channel state is not available at transmitter. We utilize the notion of estimation error outage and for the high average channel SNR limit, we show that the estimation error outage probability has a diversity order d. That means the error outage decreases with d-th power of the SNR when the SNR grows unbounded.

II. System Model

The following scalar complex Gauss-Markov model is considered.

$$\begin{aligned} s(n+1) &= \rho s(n) + u(n),\ n \geq 1,\ s(0) \sim \mathcal{CN}(0, P(0)) \\ \mathbf{z}(n) &= \mathbf{h}(n)s(n) + \mathbf{v}(n) \end{aligned} \tag{1}$$

where $u(n)$ ($\mathbf{v}(n)$) is white circularly symmetric complex Gaussian random variable (vector) with variance (covariance matrix) σ_u^2 ($\mathbf{V} = \text{diag}(V_1, V_2, \cdots, V_d)$). Consider $\mathbf{h}(n) = [h_1(n), h_2(n), \cdots, h_d(n)]^T$ to be a circularly symmetric complex Gaussian random vector of size d, with independent entries. Such a signal model characterizes e.g. measurements of a first-order Gauss-Markov process sent over d parallel independent fading channels. As previously stated, we also assume that the transmission method is uncoded analog (continuous amplitude). The objective at the decoder is that given the channel outputs, an optimal estimate of the signal $s(n)$ is calculated. This may be achieved using the Kalman filter. It is possible to show that the instantaneous estimation error variance, which we call $P(n) = E[(s(n) - \hat{s}(n))^2]$, can be written in a recursive manner in terms of its previous value and $\mathbf{h}(n)$ as

$$P(n) = \frac{\rho^2 P(n-1) + \sigma_u^2}{1 + \gamma(n)\left(\rho^2 P(n-1) + \sigma_u^2\right)} \tag{2}$$

where in our case $\gamma(n) = \sum_{i=1}^{d} |h_i(n)|^2 / V_i$ corresponds to the instantaneous channel quality. The objective is to find the asymptotic estimation error outage probability (EOP), i.e.

$$P_o(x_{\text{th}}) = \lim_{n \to \infty} \Pr(P(n) \geq x_{\text{th}}) \tag{3}$$

III. Statistical properties of estimation error variance and effect of diversity

We show that the asymptotic probability density function of $P(n)$ (given in (2)), namely $f^P(x)$ may be obtained as

$$f^P(x) = \begin{cases} \frac{1}{x^2}\int_0^{X_m} f^\gamma\left(\frac{1}{x}-\frac{1}{\rho^2 y+\sigma_u^2}\right) f^P(y)dy, & x \leqslant \sigma_u^2 \\ \frac{1}{x^2}\int_{\frac{x-\sigma_u^2}{\rho^2}}^{X_m} f^\gamma\left(\frac{1}{x}-\frac{1}{\rho^2 y+\sigma_u^2}\right) f^P(y)dy, & x > \sigma_u^2. \end{cases} \tag{4}$$

where $f^\gamma(\gamma)$ is the pdf of $\gamma(n)$. We focus on the important case of d-path receive diversity Rayleigh fading channel. We make the assumption that all the channels have the same average channel SNR, i.e. if we take $\gamma_i(n) = \frac{|h_i(n)|^2}{V_i}$, we should have $f^{\gamma_i}(\gamma_i) = \lambda e^{-\lambda\gamma_i}\mathcal{U}(\gamma_i)$, where $E(\gamma_i(n)) = \frac{1}{\lambda}$ and therefore $E(\gamma(n)) = \frac{d}{\lambda}$. In that case, the pdf of $\gamma(n)$ is χ^2-distributed and follows $f^\gamma(\gamma) = \frac{\lambda^d}{(d-1)!}\gamma^{d-1}e^{-\lambda\gamma}\mathcal{U}(\gamma)$. Note that with these definition, average stronger channels yield smaller values for λ and vice versa. Also, unless stated otherwise, SNR denotes the average channel signal to noise ratio. In this case, we can also show that for $x \leqslant \sigma_u^2$ we have that

$$f^P(x) = \frac{\lambda^d}{(d-1)!}\frac{1}{x^2}\exp\left(\frac{-\lambda}{x}\right)\sum_{i=1}^{d-1}(-1)^i \binom{d-1}{i} \kappa_i \frac{1}{x^{d-1-i}} \tag{5}$$

with

$$\kappa_i = \int_0^{X_m}\left(\frac{1}{\rho^2 y+\sigma_u^2}\right)^i \exp\left(\frac{\lambda}{\rho^2 y+\sigma_u^2}\right) f^P(y)dy, \tag{6}$$

And consequently the outage probability as in *Lemma 1*.

Lemma 1: The estimation error variance outage for any $x_{\text{th}} \leqslant \sigma_u^2$ is obtained from

$$P_o(\lambda, x_{\text{th}}) = 1-\exp\left(\frac{-\lambda}{x_{\text{th}}}\right)\sum_{i=0}^{d-1}\kappa_i\frac{(-1)^i\lambda^i}{i!}\sum_{k=0}^{d-1-i}\frac{(\frac{\lambda}{x_{\text{th}}})^k}{k!} \tag{7}$$

We define diversity order as

$$\text{DivO} = -\lim_{\text{SNR}\to\infty}\frac{\log P_o(\text{SNR}, x_{\text{th}})}{\log(\text{SNR})}. \tag{8}$$

It is also easy to show that in our case that is equal to having $P_o(\lambda, x_{\text{th}}) = \mathcal{O}(\lambda^d)$. We show that the outage probability indeed has a diversity order of d in our work as well. For that we prove Theorems I, II, III.

Theorem I: Any error function $P_o(\lambda, x_{\text{th}})$ that has the following form gives diversity order $\text{DivO} = d$

$$P_o(\lambda, x_{\text{th}}) = 1-\exp\left(-\frac{\lambda}{x_{\text{th}}}\right)\left(\sum_{i=0}^{d-1}a_i\lambda^i + \mathcal{O}(\lambda^d)\right), \tag{9}$$

given that $a_i = \frac{1}{i!x_{\text{th}}^i}$,.

Theorem II: For the high SNR regime ($\lambda \to 0$), the Taylor's series expansion of κ_i in (6) is given by

$$\kappa_i = \sum_{l=0}^{\infty}\frac{\lambda^l}{(\sigma_u^2)^{i+l}l!} \tag{10}$$

Theorem III: For the high SNR regime ($\lambda \to 0$), the expected error outage probability has diversity order $\text{DivO} = d$, i.e.

$$\lim_{\lambda\to 0}\frac{\log P_o(\lambda, x_{\text{th}})}{\log(\lambda)} = -d \quad (0 < x_{\text{th}} \leqslant \sigma_u^2) \tag{11}$$

Proof: Combines the definition for outage probability and Theorems I, II.

Theorem III shows the connection between the notion of diversity in the conventional sense of detection over wireless fading channel analysis and in the sense of estimation of scalar Gauss-Markov sources over fading channels. In other words, Theorem III means that while the probability of detection for independent signal samples achieves a diversity order of d if d independent fading channels are used, the outage probability for optimal MMSE estimator (Kalman filter) for correlated Gauss-Markov shows the same behavior, at least for the range of $0 < x_{\text{th}} \leqslant \sigma_u^2$.

IV. Conclusions

In this paper we studied the pdf of the instantaneous estimation error variance resulting from sending a scalar Gauss-Markov process over d parallel independent Rayleigh fading channel. In particular, we focused on the effect of diversity on the performance of the Kalman filter, in terms of the outage for the instantaneous estimation error variance. The pdf of the instantaneous estimation error variance takes the form of a two-part function, where the first part is then characterized by a closed-form solution. The outage is also given using a closed-form formula for the first part. Furthermore, we showed that in the limit of the high channel SNR, the outage probability achieves a diversity order the same as the number of channels.

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