Distributed Approach to the Optimal Power Flow Problem

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***Abstract*—We address the Optimal Power Flow (OPF) problem, which is NP hard in general. The OPF problem plays a key role in efficient control of electrical networks. Solving the problem not only in transmission networks, but also in distribution networks is challenging because of the constant growth of the smart grid. As a result, related algorithms must possess rich scalability properties to keep up with the network growth. To address these issues, we develop a distributed algorithm for the OPF problem by decomposing the problem among the buses of the network. The proposed algorithm is based on alternating direction method of multipliers and sequential approximation techniques. The original problem is equivalently split into small local subproblems (one for each bus), which are coordinated via thin communication protocol, enriching the algorithm with scalability properties.**

***Index Terms*— Optimal power flow, distributed optimization, smart grid, interactive power networks.**

I. Introduction

The optimal power flow (OPF) problem in electrical networks determines, the amount of power to generated at each generation point and how to dispatch the power. A global network-wide objective criterion is optimized, while ensuring that the power demand of each consumer is met and that the related laws of physics are not violated.

The problem was originally presented by Carpentier in the sixties [1], and has been extensively studied since then. It become of great importance in efficient operation of power systems [2]. The problem is shown to be NP-hard, see [3], and therefore practical and general purpose algorithms must rely on some approximations or heuristics.

In this paper, we first present a distributed algorithm for the *general* OPF problem. We do not rely on SDP relaxation, and therefore our approach is not restricted to any special classes of networks. We capitalize on alternating direction method of multipliers (ADMM) [4] to design a distributed algorithm among electrical network buses. The original problem is split into subproblems (one for every bus), which are coordinated via a light protocol to compute a desirable feasible point. In the case of optimization subproblems at electrical network buses, we capitalize on sequential approximations, in order to gracefully manage the nonconvexity issues.

II. System model and Problem Formulation

Consider an electrical network with N buses where $\mathcal{N}=\{1,2,\dots,N\}$ is the set of buses and $\mathcal{L}$ $(\subseteq \mathcal{N}\times\mathcal{N})$ is the set of flow lines. We denote by $i_k=i_k^{\text{Re}}+ji_k^{\text{Im}}$ the current injection and by $v_k=v_k^{\text{Re}}+jv_k^{\text{Im}}$ the voltage at bus $k\in\mathcal{N}$. Let $p_k^{\text{D}}+jq_k^{\text{D}}\in\mathbb{C}$ and $p_k^{\text{G}}+jq_k^{\text{G}}\in\mathbb{C}$ denote the complex power demand and the complex power generated by bus $k\in\mathcal{N}$, respectively. Thus, the complex power $p_k+jq_k\in\mathbb{C}$ injected to bus k is given by $p_k+jq_k=(p_k^{\text{G}}+jq_k^{\text{G}})-(p_k^{\text{D}}+jq_k^{\text{D}})$.

For clarity, we let $\mathbf{p}^{\text{G}}$, $\mathbf{q}^{\text{G}}$, $\mathbf{p}^{\text{D}}$, $\mathbf{q}^{\text{D}}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{i}$, $\mathbf{i}^{\text{Re}}$, $\mathbf{i}^{\text{Im}}$, $\mathbf{v}$, $\mathbf{v}^{\text{Re}}$, and $\mathbf{v}^{\text{Im}}$ denote the vectors $(p_k^{\text{G}})_{k\in\mathcal{N}}$, $(q_k^{\text{G}})_{k\in\mathcal{N}}$, $(p_k^{\text{D}})_{k\in\mathcal{N}}$, $(q_k^{\text{D}})_{k\in\mathcal{N}}$, $(p_k)_{k\in\mathcal{N}}$, $(q_k)_{k\in\mathcal{N}}$, $(i_k)_{k\in\mathcal{N}}$, $(i_k^{\text{Re}})_{k\in\mathcal{N}}$, $(i_k^{\text{Im}})_{k\in\mathcal{N}}$, $(v_k)_{k\in\mathcal{N}}$, $(v_k^{\text{Re}})_{k\in\mathcal{N}}$, and $(v_k^{\text{Im}})_{k\in\mathcal{N}}$, respectively. We denote by $i_{nm}^{\text{Re}}+ji_{nm}^{\text{Im}}\in\mathbb{C}$ the complex current and by $p_{nm}+jq_{nm}\in\mathbb{C}$ the complex power transferred from bus n to the rest of the network through the flow line $(n,m)\in\mathcal{L}$. The admittance matrix $\mathbf{Y}\in\mathbb{C}^{N\times N}$ of the network is given by

$$\mathbf{Y}=\begin{cases} y_{nn}+\sum_{(n,l)\in\mathcal{L}} y_{nl}, & \text{if } n=m,\\ -y_{nm}, & \text{if } (n,m)\in\mathcal{L}\ ,\\ 0, & \text{otherwise,}\end{cases} \tag{1}$$

where $y_{nm}=g_{nm}+jb_{nm}\in\mathbb{C}$ is the admittance in the flow line $(n,m)\in\mathcal{L}$, and $y_{nn}=g_{nn}+jb_{nn}\in\mathbb{C}$ is the admittance to ground at bus n. We let $\mathbf{G}\in\mathbb{R}^{N\times N}$ and $\mathbf{B}\in\mathbb{R}^{N\times N}$ denote the real and imaginary parts of $\mathbf{Y}$, respectively, i.e., $[\mathbf{G}]_{nm}=g_{nm}$ and $[\mathbf{B}]_{nm}=b_{nm}$ yielding $\mathbf{Y}=\mathbf{G}+j\mathbf{B}$. Finally, we let $\mathbf{c}_{nm}=(g_{nm},-g_{nm},b_{nm},-b_{nm})$ and $\mathbf{d}_{nm}=(b_{nm},-b_{nm},-g_{nm},g_{nm})$ for notational simplicity.

A. Centralized formulation

We denote by f_k^{G} the cost of generating power at bus $k\in\mathcal{G}$, where $\mathcal{G}\subseteq\mathcal{N}$ denotes the set of generator buses. We can now express formally the OPF problem as

$$\begin{aligned}
\text{minimize}\quad & \sum_{k\in\mathcal{G}} f_k^{\text{G}}(p_k^{\text{G}}) && \text{(2a)}\\
\text{subject to}\quad & \mathbf{i}^{\text{Re}}+j\mathbf{i}^{\text{Im}}=\mathbf{G}\mathbf{v}^{\text{Re}}-\mathbf{B}\mathbf{v}^{\text{Im}}+j\left(\mathbf{B}\mathbf{v}^{\text{Re}}+\mathbf{G}\mathbf{v}^{\text{Im}}\right), && \text{(2b)}\\
& p_k+jq_k=p_k^{\text{G}}-p_k^{\text{D}}+j\left(q_k^{\text{G}}-q_k^{\text{D}}\right),\ k\in\mathcal{N}, && \text{(2c)}\\
& i_{nm}^{\text{Re}}+ji_{nm}^{\text{Im}}=\left(\mathbf{c}_{nm}^{\text{T}}+j\mathbf{d}_{nm}^{\text{T}}\right)(v_n^{\text{Re}},v_m^{\text{Re}},v_n^{\text{Im}},v_m^{\text{Im}}),\ (n,m)\in\mathcal{L}, && \text{(2d)}\\
& \mathbf{p}+j\mathbf{q}=\mathbf{v}^{\text{Re}}\bullet\mathbf{i}^{\text{Re}}+\mathbf{v}^{\text{Im}}\bullet\mathbf{i}^{\text{Im}}+j\left(\mathbf{v}^{\text{Im}}\bullet\mathbf{i}^{\text{Re}}-\mathbf{v}^{\text{Re}}\bullet\mathbf{i}^{\text{Im}}\right), && \text{(2e)}\\
& p_{nm}+jq_{nm}=v_n^{\text{Re}}i_{nm}^{\text{Re}}+v_n^{\text{Im}}i_{nm}^{\text{Im}}+j\left(v_n^{\text{Im}}i_{nm}^{\text{Re}}-v_n^{\text{Re}}i_{nm}^{\text{Im}}\right),\ (n,m)\in\mathcal{L}, && \text{(2f)}\\
& p_k^{\text{G,min}}\le p_k^{\text{G}}\le p_k^{\text{G,max}},\ k\in\mathcal{N}, && \text{(2g)}\\
& q_k^{\text{G,min}}\le q_k^{\text{G}}\le q_k^{\text{G,max}},\ k\in\mathcal{N}, && \text{(2h)}\\
& (i_{nm}^{\text{Re}})^2+(i_{nm}^{\text{Im}})^2\le(i_{nm}^{\text{max}})^2,\ (n,m)\in\mathcal{L}, && \text{(2i)}\\
& p_{nm}^2+q_{nm}^2\le(s_{nm}^{\text{max}})^2,\ (n,m)\in\mathcal{L}, && \text{(2j)}\\
& |p_{nm}|\le p_{nm}^{\text{max}},\ (n,m)\in\mathcal{L}, && \text{(2k)}\\
& (v_k^{\text{min}})^2\le(v_k^{\text{Re}})^2+(v_k^{\text{Im}})^2\le(v_k^{\text{max}})^2,\ k\in\mathcal{N}, && \text{(2l)}
\end{aligned}$$

where the variables are $\mathbf{p}^{\text{G}}$, $\mathbf{q}^{\text{G}}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{i}^{\text{Re}}$, $\mathbf{i}^{\text{Im}}$, $\mathbf{v}^{\text{Re}}$, $\mathbf{v}^{\text{Im}}$, and i_{nm}^{Re}, i_{nm}^{Im}, p_{nm}, q_{nm} for $(n,m)\in\mathcal{L}$.

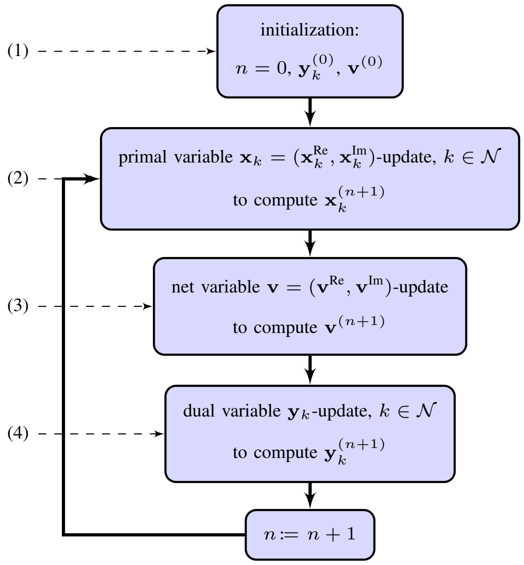

Fig. 1: *Algorithm 1*

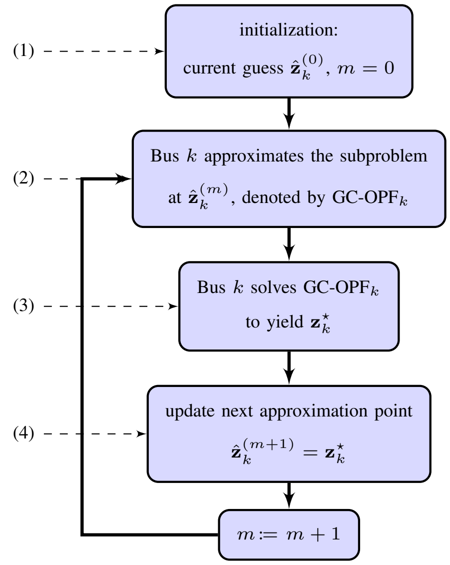

Fig. 2: *Algorithm 2*

B. General Consensus Form

By carefully identifying the coupling variables of problem (2), we can equivalently formulate the problem in the form of a general consensus problem [4, § 7.2], where fully decentralized implementation can be realized. To briefly formalize the idea above, we first denote by $\mathcal{N}_k$ the set of bus k itself and its neighboring buses, i.e. $\mathcal{N}_k = \{k\} \cup \{n | (k,n) \in \mathcal{L}\}$. Copies of real and imaginary parts of the voltages corresponding to buses in $\mathcal{N}_k$ is denoted by $\mathbf{x}_k^{\text{Re}} \in \mathbb{R}^{|\mathcal{N}_k|}$ and $\mathbf{x}_k^{\text{Im}} \in \mathbb{R}^{|\mathcal{N}_k|}$, respectively. For notational convenience, we let $(\mathbf{x}_k^{\text{Re}})_1 = v_k^{\text{Re}}$ and $(\mathbf{x}_k^{\text{Re}})_1 = v_k^{\text{Im}}$. We refer to $\mathbf{v}^{\text{Re}}$ and $\mathbf{v}^{\text{Im}}$ as *real and imaginary net variables*, respectively. Note that the copies of net variable v_k^{Re} and v_k^{Im} are shared among $|\mathcal{N}_k|$ entities, which we call the degree of net variable v_k^{Re} or v_k^{Im}. The coupling of variables is imposed by

$$\mathbf{x}_k^{\text{Re}} = \mathbf{E}_k \mathbf{v}^{\text{Re}}, \qquad \mathbf{x}_k^{\text{Im}} = \mathbf{E}_k \mathbf{v}^{\text{Im}}, \text{ where} \tag{3}$$

$$(\mathbf{E}_k)_{nm} = \begin{cases} 1 & \text{if } (\mathbf{x}_k^{\text{Re}})_n \text{ is a local copy of } v_m^{\text{Re}} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

By using (3) and the local variables $\mathbf{z}_k = (p_k^{\text{G}}, q_k^{\text{G}}, p_k, q_k, i_k^{\text{Re}}, i_k^{\text{Im}}, \mathbf{x}_k^{\text{Re}}, \mathbf{x}_k^{\text{Im}}, \bar{\mathbf{i}}_k^{\text{Re}}, \bar{\mathbf{i}}_k^{\text{Im}}, \bar{\mathbf{p}}_k, \bar{\mathbf{q}}_k)$ associated with every bus k, the equivalent general consensus problem for OPF (GC-OPF) is obtained. We denote by $\mathbf{y}_k$ the dual variables associated with constraints (3) of the resulting GC-OPF problem. Standard ADMM techniques can then be readily applied to GC-OPF problem [4, § 7.2], see Figure 1.

Note that step 3 and 4 of the algorithm can be solved in a straightforward manner. In contrast, step 2 is the most challenging iteration, where the associated subproblems (one for each bus) are NP-hard. However, for efficient implementation of the algorithm, polynomial-time algorithms are desirable, even with a loss in the optimality. Therefore, we capitalize on sequential convex approximations to design a good heuristic. Our approach is inspired from the approximations used in [5], in the context of centralized OPF. The key idea is to use first order Taylor's approximations in an iterative manner to convexify the nonconvex constraints of GC-OPF problem. Figure 2 shows a block diagram of the proposed algorithm.

III. Properties of the distributed solution method

Because the original problem (2) is nonconvex and is NP-hard, optimality and convergence guarantees of non-global methods are usually difficult to achieve if not impossible. Nevertheless, we investigate some of the optimality properties of our proposed *Algorithm 1* and *Algorithm 2*. Roughly speaking, we show that the results of *Algorithm 2* satisfy Karush-Kuhn-Tucker (KKT) conditions for the original problem to be solved at step 2 of *Algorithm 1*. This results is later used to characterize the solutions of *Algorithm 1*. These properties of the proposed algorithms are numerically substantiated. Details are skipped due to page limitations.

IV. Conclusions

We proposed a distributed algorithm for the optimal power flow problem by decomposing it among the buses. Only a thin communication protocol among neighboring buses is required during the algorithm, resulting in rich scalability properties. The proposed algorithms were based on alternating direction method of multipliers and sequential convex approximations techniques. The optimality properties of the proposed algorithms were investigated, under mild conditions and were numerically substantiated.

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